ON THE CRITICAL EXPONENTS OF NORMS IN AN n-DIMENSIONAL SPACE

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UDC 517.948.3

The natural number q is called the critical exponent (see [1]) of the norm in the n-dimensional space E^n (real or complex), if:

1) for all linear operators A in E^n with $\|A\| = 1$

$$\|A^q\| = 1 \Rightarrow \|A^m\| = 1 \quad (m > q);$$

2) there exists a linear operator A_0 ($\|A_0\| = 1$) such that $\|A_0^{q-1}\| = 1$, $\|A_0^q\| < 1$.

Let us denote by $\rho(A)$ the spectral radius (the largest of the absolute values of the eigenvalues) of the operator A. It is known that

$$\rho(A) = \inf_m \sqrt[m]{\|A^m\|} = \lim_{m\to\infty} \sqrt[m]{\|A^m\|}.$$

If $\|A\| = 1$, then $\|A^m\| \leq 1$ ($m = 1, 2, \ldots$), $\rho(A) \leq 1$ and the inequality $\rho(A) < 1$ is equivalent to the existence of an exponent m such that $\|A^m\| < 1$. The smallest of these exponents will be denoted by q(A). The critical exponent defined above is equal to $\sup_{\|A\|=1,\, \rho(A)<1} q(A)$, provided that the supremum exists. If the supremum is infinite, then the critical exponent does not exist.

The first critical exponent has been introduced and computed in [2] for the arithmetic space in the special case of the c-norm

$$\|x\| = \max_{1\leq k\leq n} |x_k| \quad (x = (x_1, \ldots, x_n)),$$

being equal to $n^2 - n + 1$ (see also [3]).

Let us consider in the arithmetic space the l_p-norm

$$\|x\| = \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} (1 \leqslant p < \infty), \quad \|x\| = \max_{1\leq k\leq n} |x_k| \quad (p = \infty).$$

The critical exponents for $p = 1$ and $p = \infty$ are equal, i.e., $n^2 - n + 1$, for $p = 2$ is equal to n [4], while for the remaining values of p the problem of its existence is still open, except for some particular integers for which the existence of the critical exponent has been announced in [5].

Yu. I. Lyubich has stated the assumption that if the unit sphere is an analytic variety, then the critical exponent exists. This question is open; however, we have the following theorem.

THEOREM 1. If the unit sphere $S = \{x : \|x\| = 1\}$ can be embedded in an algebraic variety F not containing the origin, then the critical exponent exists.

Thus, if the sphere is an algebraic variety or even a piecewise-algebraic variety whose pieces by continuation do not pass through the origin, then the critical exponent exists. In particular, the critical exponent exists for every polytope. The last fact has been obtained by Perles from other considerations [5]. Besides, it should be noted that Perles succeeded in obtaining a better (in some cases a more exact) estimate for the critical exponent, while the estimate which follows from the proof of Theorem 1 is strongly excessive.

Translated from Sibirskii Matematicheskii Zhurnal, Vol. 12, No. 3, pp. 672-675, May-June, 1971. Original article submitted January 14, 1970.

Proof. Without loss of generality we can assume that the variety F is defined by a single equation $P(x) \equiv P(x_1, \ldots, x_n) = 0$, $P(0) \neq 0$ (otherwise it is sufficient to extend the variety, leaving some single nonhomogeneous equation).

Let us associate to the linear operator A a sequence of sets $R_k(A) \subset E^n$ ($k = 0, 1, 2, \ldots$):

$$R_k(A) = \{x \mid x \in E^n, \quad P(x) = P(Ax) = \ldots = P(A^k x) = 0\}.$$

Obviously, $R_k \subset R_{k-1}$, $AR_k \subset R_{k-1}$ ($k \geq 1$) and

$$R_s = R_{s-1} \Rightarrow R_{s+1} = R_s. \tag{1}$$

$R_k(A)$ is the set of the solutions of the system

$$f_j(x) = P(A^j x) = 0 \quad (j = 0, 1, \ldots, k). \tag{2}$$

The functions $f_j(x)$ which appear here are polynomials and $\deg f_j = \deg P$.

We denote by N the dimension of the linear space of all polynomials with n variables of degree not exceeding deg P. Obviously, there exists $\tilde{q} \leq N-1$ such that the polynomial $f_{\tilde{q}+1}$ is a linear combination of the polynomials f_j ($j \leq \tilde{q}$). This means that $R_{\tilde{q}+1}(A) = R_{\tilde{q}}(A)$. Thus, for each A there exists a natural number $\tilde{q} = \tilde{q}(A) \leq N-1$ such that $R_{\tilde{q}+1}(A) = R_{\tilde{q}}(A)$, whence, making use of (1), we have for all A

$$R_m(A) = R_{m-1}(A) \quad (m \geq N). \tag{3}$$

Let $\|A\| = 1$, $\|A^{N-1}\| = 1$. Then the unit vector for which $\|A^{N-1}\| = \max_{\|x\|=1} \|A^{N-1}x\|$ is attained belongs to $R_{N-1}(A)$, whence, by virtue of (3),

$$P(A^m x) = 0 \quad (m \geq N-1). \tag{4}$$

If $\rho(A) < 1$, then $\lim_{m \to \infty} A^m = 0$ and from (4) it follows that $P(0) = 0$, i.e., $0 \in F$. Therefore $\rho(A) = 1$, i.e., $\|A^m\| = 1$ ($m \geq N-1$).

Thus, the existence of the critical exponent is proved and it is smaller or equal to $N-1$.

COROLLARY 1. If in a real space all the points of the unit sphere satisfy the equation $P(|x_1|, |x_2|, \ldots, |x_n|) = 0$, where P is a nonhomogeneous polynomial of n variables, then the critical exponent exists.

Indeed, we can take as the variety F the union of the algebraic varieties $M\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$ (the ε_i's take on independently the values 1 and −1), defined by the equations $P(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n) = 0$.

COROLLARY 2. In the real arithmetic space with the l_p norm with p rational, the critical exponent exists.

For integer p this is obvious by virtue of Corollary 1.

Let $p = s/r$ (s and r are relatively prime integers). Let us write the equation of the unit sphere S in the form

$$\sum_{i=1}^{n} |x_i|^{s/r} = 1. \tag{5}$$

Denote $z_i^r = |x_i|^s$. Equation (5) can be replaced by an equivalent system of equations

$$\begin{cases} g_i = z_i^r - |x_i|^s = 0 \quad (i = 1, \ldots, n), \\ \varphi_0 = \sum_{i=1}^{n} |z_i| - 1 = 0. \end{cases}$$

We define by induction the polynomial $\varphi_k(|x_1|, \ldots, |x_k|, z_{k+1}, \ldots, z_n)$ ($k = 1, \ldots, n$) as the resultant of the two polynomials φ_{k-1} and g_k with respect to the variable z_k.

Let us prove that φ_k is nonhomogeneous, i.e., $\varphi_k(0) \neq 0$. For φ_0 this is obvious. The polynomial φ_k has the form

$$\varphi_k(|x_1|, \ldots, |x_k|, z_{k+1}, \ldots, z_n) = \prod_j \varphi_{k-1}(|x_1|, \ldots, |x_{k-1}|, \alpha_j(|x_k|), z_{k+1}, \ldots, z_n),$$

where $\alpha_j(|x_k|) = |x_k|^{s/r} \exp(i 2\pi j/r)$, ($j = 0, 1, \ldots, r-1$). Since $\alpha_j(0) = 0$, we have $\varphi_k(0, \ldots, 0) = \prod \varphi_{k-1}(0, \ldots, 0) \neq 0$ by induction.

Let us note that all points of the unit sphere S satisfy the equation $\varphi_n(|x_1|, \ldots, |x_n|) = 0$.

Similarly one proves the following corollary.

COROLLARY 3. In the complex arithmetic space with the l_p norm with p rational, the critical exponent exists.

Let us remark that in Theorem 1 the requirement $0 \notin F$ is essential:

Example. Assume that the unit sphere S of the norm in a two-dimensional real arithmetic space is defined by the equations

$$x_2 = \begin{cases} x_1^3, & 1 \leqslant |x_1| \leqslant 2, \\ 3x_1 + 2, & -2 \leqslant x_1 \leqslant -1, \\ 3x_1 - 2, & 1 \leqslant x_1 \leqslant 2. \end{cases}$$

The smallest algebraic variety F_0 containing the sphere S, is the union of the algebraic curves $x_2 = x_1^3$, $x_2 = 3x_1 + 2$ and $x_2 = 3x_1 - 2$. Since $0 \in F_0$, the conditions of Theorem 1 cannot be satisfied.

On the other hand, consider the sequence of linear operators A_k (k = 1, 2, . . .), defined by the diagonal matrices $A_k = \operatorname{diag}\{2^{-1/k}, 2^{-1/k}\}$. It is easy to check that $\|A_k\| = \|A_k^k\| = 1$ and $\|A_k^{k+1}\| < 1$, i.e., the critical exponent does not exist.

In conclusion, let us consider the similar problem for finite-dimensional associative normed algebras (real or complex). The norm of the algebra has to possess the additional property

$$\|AB\| \leqslant \|A\| \|B\|. \tag{6}$$

The critical exponent is defined in the same way as before, except that the elements of the algebra occur in the definition instead of the linear operators.

As an example we have the algebra of square matrices of order n, normed with any matrix norm [6], for example

$$\|A\| = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2} = \sqrt{\operatorname{Sp}(A^*A)}.$$

THEOREM 2. If the unit sphere S of the normed algebra can be embedded in an algebraic variety F not containing the origin, then the critical exponent exists.

Proof. To each element A of the algebra we associate the linear operator Φ_A of multiplication by A on the left. Obviously

$$\|\Phi_A\| \leqslant \|A\|, \tag{7}$$

$$\|A^k\| = \|A^{k+1}\| \Rightarrow \|\Phi_A^{k+1}\| = \|A^{k+1}\|. \tag{8}$$

By virtue of Theorem 1 there exists a natural number q such that

$$\|\Phi\| = \|\Phi^q\| = 1 \Rightarrow \|\Phi^m\| = 1 \quad (m > q) \tag{9}$$

for all linear operators Φ.

Let $\|A\| = \|A^{q+1}\| = 1$. Then, by virtue of (8), $\|\Phi_A\| = \|\Phi_A^q\| = 1$ and by (9) $\|\Phi_A^m\| = 1$ $(m > q)$. But according to (7), $\|\Phi_A^m\| \leq \|A^m\|$, whence $\|A^m\| = 1$ $(m > q + 1)$. This proves the theorem.

The authors wish to thank Yu. I. Lyubich for his interest in the paper.

LITERATURE CITED

1. V. Ptak, "Critical exponents," Proc. Colloq. Convexity, Copenhagen, 244-248 (1965).
2. J. Marik and V. Ptak, "Norms, spectra and combinatorial properties of matrices," Czech. Math. Journal, 10, No. 2, 181-196 (1960).
3. Yu. I. Lyubich and M. I. Tabachnikov, "On a theorem of Marik – Ptak," Sibirsk. Matem. Zh., 10, No. 2, 470-473 (1969).
4. V. Ptak, "Norms and the spectral radius of matrices," Czech. Math. Journal, 12, No. 4, 555-557 (1962).

5. M. A. Perles, "Critical exponents of convex sets," Proc. Colloq. Convexity, Copenhagen, 221-228 (1965).
6. D. K. Faddeev and V. N. Faddeeva, Computational Methods of Linear Algebra [in Russian], Fizmatgiz, Moscow – Leningrad (1963).